Integral of a complex-valued function.

$f : [a,b] \to \mathbb{C} = x(t) + i y(t)$

$$\int_a^b f(t)\,dt := \int_a^b x(t)\,dt + i\int_a^b y(t)\,dt.$$

Linear: $\int_a^b (\alpha f(t) + \beta g(t))\,dt = \alpha\int_a^b f(t)\,dt + \beta\int_a^b g(t)\,dt \quad (\alpha, \beta \in \mathbb{C}).$

Additive: $\int_a^b f(t)\,dt + \int_b^c f(t)\,dt = \int_a^c f(t)\,dt$

Change of variables: $t \to g(t)$ piecewise differentiable, increasing on $[c,d]$, $g(c) = a$, $g(d) = b$: $\int_a^b f(t)\,dt = \int_c^d f(g(s))\,g'(s)\,ds$

Change of orientation: $\int_a^b f(t)\,dt = \int_a^b f(a+b-t)\,dt$

Lemma. $\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt$

Remark: $f : [a,b] \to \mathbb{R}$: use $-|f| \le f \le |f|$.

Proof. Change of phase trick.

$M := \int_a^b f(t)\,dt$

$$|M|^2 = \bar{M}\int_a^b f(t)\,dt = \operatorname{Re} \bar{M}\int_a^b f(t)\,dt = \int_a^b \operatorname{Re} \bar{M} f(t)\,dt \le$$

$$\int_a^b |\bar{M}|\,|f(t)|\,dt \le |M|\int_a^b |f(t)|\,dt \Rightarrow$$

$|M| \le \int_a^b |f(t)|\,dt$ if $M \ne 0$

($M = 0$ – obvious) ∎

Line (contour) integral.

Let γ be a piece-wise smooth curve. f – a function continuous on $\gamma[a,b]$.

$z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $t \in [a,b]$ – parameterization.

Def. (Line integrals).

$$\oint_\gamma f(z)\,dx := \int_a^b f(z(t))\,x'(t)\,dt$$

$$\oint_\gamma f(z)\,dy := \int_a^b f(z(t))\,y'(t)\,dt$$

$$\oint f(z)\,dz := \int^b f(z(t))\,z'(t)\,dt \quad \oint f(z)\,dz = \int f(z)\,dz$$

$$\oint_\gamma f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt = \oint_\gamma f(z)\,dx + i\int_\gamma f(z)\,dy.$$

Properties. 1) Independent of parametrization.

$\gamma : [a,b] \to \mathbb{C}$, $s : [c,d] \to [a,b]$ - increasing, piecewise-differentiable. $s(c)=a,\ s(d)=b$

Then $\oint_\gamma f(z)\,dz = \int_a^b f(z(s))\,z'(s)\,ds = \int_c^d f(z(s(t)))\,z'(s)\,s'(t)\,dt =$

$$\int_c^d f(z(s(t)))\,\big(z(s(t))\big)'\,dt.$$

2) Change of orientation: $s(c)=b,\ s(d)=a$

$\gamma : [a,b] \to \mathbb{C}$, $s : [c,d] \to [a,b]$ - decreasing, piecewise-differentiable.

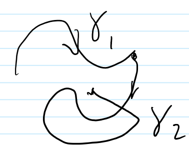
Then $\oint_\gamma f(z)\,dz = \int_a^b f(z(s))\,z'(s)\,ds = \int_d^c f(z(s(t)))\,z'(s)\,s'(t)\,dt =$

$$-\int_c^d f(z(s(t)))\,\big(z(s(t))\big)'\,dt.$$

Notation: γ^- $(-\gamma)$ - γ traversed in the opposite direction.

3) Additivity: $\gamma_1 : [a,b] \to \mathbb{C}$, $\gamma_2 : [b,c] \to \mathbb{C}$, $\gamma_1(b) = \gamma_2(b)$.

Define: $(\gamma_1+\gamma_2)(t) = \begin{cases} \gamma_1(t), & a \le t \le b \\ \gamma_2(t), & b \le t \le c \end{cases}$ on $[a,c]$.



$\gamma + (-\gamma) \ne 0$

$$\oint_{\gamma_1+\gamma_2} f(z)\,dz = \oint_{\gamma_1} f(z)\,dz + \oint_{\gamma_2} f(z)\,dz$$ - by direct computation.

4) Linearity $\alpha, \beta \in \mathbb{C}$ $\oint_\gamma (\alpha f(z) + \beta g(z))\,dz = \alpha\oint_\gamma f(z)\,dz + \beta\oint_\gamma g(z)\,dz.$

A very important example.

$\gamma : [0, 2\pi] \to \mathbb{C}$ $\quad z(t) = a + re^{it}$.

$n \in \mathbb{Z}$: $\oint_\gamma (z-a)^n\,dz = \int_0^{2\pi} r^n e^{int}\, rie^{it}\,dt = ir^{n+1}\int_0^{2\pi} e^{i(n+1)t}\,dt =$

$ir^{n+1}\int_0^{2\pi}\big(\cos(n+1)t + i\sin(n+1)t\big)\,dt = \begin{cases} ir^{n+1}\frac{1}{n+1}\big(\sin(n+1)t - i\cos(n+1)t\big)\Big|_0^{2\pi} = 0, & n \ne -1 \\ 2\pi i, & n = -1. \end{cases}$

$$\boxed{\oint_\gamma (z-a)^n\,dz = \begin{cases} 2\pi i, & n=-1 \\ 0, & n \ne -1. \end{cases}}$$

$$(z-a)^n = \left(\frac{(z-a)^{n+1}}{n+1}\right)' \quad n \ne -1$$

$\gamma \qquad \{ \; 0, \; n \neq -1.$

$(z-a)^n = \left(\dfrac{\cdots}{n+1} \right)' \quad n \neq -1$

Heuristically. $(z-a)^{-1} = \left(\underbrace{\log(z-a)} \right)'$

$\int (z-a)^{-1} = 2\pi i - 0$